Synthesis of MITE Log-Domain Filters with Unique Operating Points

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***Abstract*— Practical log-domain filter circuits might have multiple operating points in regions in which the translinear element does not obey the exponential law. In this paper, a method is proposed to implement any filter by a log-domain circuit that necessarily has a unique operating point. Any state-space description of the filter is shown to have an equivalent description that can be implemented by such a circuit. This methodology is applied to the synthesis of Multiple-Input Translinear Element (MITE) filters. As an example, shifted-companion-form (SCF) filters are synthesized. Further, it is proved that the resulting filters have a unique operating point.**

I. Introduction

Log-domain filters are usually designed under the assumption that the translinear element has ideal exponential characteristics. However, this exponential characteristic is valid in only a certain region of operation of the translinear element. Hence, though the ideal equations indicate that the circuit has a unique operating point, it might happen that the filter implementation leads to multiple operating points. The existence of multiple operating points in log-domain filters using MOSFETs in the subthreshold region is reported in [1]. However, no general procedure is given to synthesize log-domain filters in a manner that avoids this phenomenon. We propose a synthesis methodology using first-order low-pass filters (FOLPFs for short). Synthesis using FOLPFs has the advantage that the exponential state-space transformation is already implicitly done in the FOLPF. Further, it will be shown that the state-space decomposition can be done such that the resulting circuit has a unique operating point.

II. Mathematical preliminaries

$\mathbb{R}^{n\times m}$ denotes the set of all real $n \times m$ matrices. The *sign pattern* of a real matrix A, denoted by $sign(A)$, is defined as the matrix obtained by replacing each element of A by its sign i.e

$$[sign(A)]_{ij} = \begin{cases} 1 & \text{if } A_{ij} > 0, \\ -1 & \text{if } A_{ij} < 0, \\ 0 & \text{if } A_{ij} = 0. \end{cases}$$

The *qualitative class* $\mathcal{Q}(A)$ of a real matrix $A \in \mathbb{R}^{n\times m}$ is defined by $\mathcal{Q}(A) = \{B \in \mathbb{R}^{n\times m} | sign(B) = sign(A)\}$. A square matrix A is a *sign-nonsingular* (SNS) matrix if every matrix in its qualitative class is nonsingular.

III. Constraints on the state-space equations

The general state-space form of any multiple-input multiple-output (MIMO) filter is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and A, B, C and D are matrices of appropriate dimensions.

Definition 1: The state-space system in (1) is said to be *implementable by FOLPFs* if A has negative diagonal entries. Clearly, this means that one can write (1) (in terms of low-pass filters) as

$$\begin{aligned} \dot{x} + Ex &= A'x + Bu \\ y &= Cx + Du \end{aligned} \tag{2}$$

where E is a diagonal matrix with positive diagonal and $A' = A + E$ has zero diagonal.

Definition 2: The state-space system in (1) is said to have a *sign-unique operating point* if A is a SNS matrix.

The motivation behind the above definition is seen in Theorem 1 in [2], a slightly modified version of which is the following:

Theorem 1: Let U be a open convex subset of $\mathbb{R}^n$ and $f : U \subseteq \mathbb{R}^n \mapsto \mathbb{R}^n$ a C^1 function such that all the elements of the Jacobian matrix $Df(x)$ of f have the same sign for all $x \in U$. Then, f is injective on U if $Df(x)$ is a SNS matrix.

It will be seen that solving for the operating point of a Multiple-Input Translinear Element (MITE) implementation of (1) requires the solution of a nonlinear equation of the form $f(V) = 0$ where $f : (0, V_{dd})^n \mapsto \mathbb{R}^n$ is such that the partial derivative $\frac{\partial f_i}{\partial x_j}$ has the same sign as A_{ij}. Hence, the operating point is unique if A is a SNS matrix. Therefore, all state-space systems will be required to have a sign-unique operating point.

A. Example: Shifted-Companion-Form Filters

The shifted-companion-form (SCF) [3] lends itself easily to synthesis by the proposed methodology. The MITE implementation of a SCF state-space system is particularly simple in the case where the transmission zeros are formed by summation

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of the state variables. The (A,B,C,D) matrices from (1) of this single-input single-output system are [3]

$$A=\begin{bmatrix} -a_{n-1}-\alpha & -a_{n-2} & -a_{n-3} & \cdots & -a_1 & -a_0 \\ 1 & -\alpha & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\alpha & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & -\alpha \end{bmatrix}, \quad (3)$$

$B=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^t$, $C=\begin{bmatrix}b_{n-1} & b_{n-2} & \cdots & b_0\end{bmatrix}$, and $D=d$ (a scalar).

The above state-space realization is obtained from the transfer function

$$\frac{Y(s)}{U(s)}=\frac{b_{n-1}(s+\alpha)^{n-1}+b_{n-2}(s+\alpha)^{n-2}+\cdots b_0}{(s+\alpha)^n+a_{n-1}(s+\alpha)^{n-1}+\cdots a_0}+d$$

Theorem 2: The shifted companion matrix A in (3) is a SNS matrix if $\alpha>0$ and $a_0,a_1\ldots a_{n-1}$ are nonnegative. If $\alpha=0$, then the companion matrix A is a SNS matrix if $a_0\neq 0$.

Proof : Let $\alpha>0$ and $a_0,a_1\ldots a_{n-1}$ be nonnegative. It suffices to show that $Mx=0\Rightarrow x=0$ for any $M\in\mathcal{Q}(A)$. $Mx=0$ implies that

$$\begin{aligned} |M_{nn-1}|x_{n-1} &= |M_{nn}|x_n \\ |M_{n-1n-2}|x_{n-2} &= |M_{n-1n-1}|x_{n-1} \\ &\vdots \\ |M_{21}|x_1 &= |M_{22}|x_2 \\ |M_{11}|x_1+|M_{12}|x_2+\cdots|M_{1n}|x_n &= 0 \end{aligned} \quad (4)$$

It should be noted that all the elements of M above (except the last row) and M_{11} are necessarily nonzero. Clearly, we have $x_i=\beta_i x_n$ for $i=1,2,\ldots n$ with $\beta_i>0$. The last equation in (4) yields $x_n\sum_{i=1}^{n}|M_{1i}|\beta_i=0$, which implies that x_n and hence x is zero. The proof is similar and easier when $\alpha=0$. □

To show that constraining A in (1) to be a SNS matrix does not restrict the set of transfer functions obtainable from (1), we prove the following result:

Theorem 3: There exists a SNS matrix J with negative diagonal entries similar to any Hurwitz matrix A. In particular, J can be written as a direct sum (i.e a block diagonal matrix) of shifted-companion matrices of the type shown in (3) (with $\alpha>0$)

Proof : J can be chosen to be the (real) Jordan canonical form of A [4]. A more useful SNS matrix similar to a Hurwitz matrix A is obtained by the following method:

Let $\alpha>0$ be such that $-\alpha>\max_{\lambda\in\sigma(A)}\Re e(\lambda)$ where $\sigma(A)$ is the set of eigenvalues of A. Define $A'=A+\alpha I$. Clearly, $\sigma(A')=\{\lambda+\alpha|\lambda\in\sigma(A)\}$ and hence, by the definition of α, A' is Hurwitz. Taking the rational form or the rational canonical form of A' [4], we obtain a matrix J' that is a direct sum of companion matrices of the form depicted in (3) (with $\alpha=0$). It follows from the assumption that A' is Hurwitz that each block companion matrix in J' is Hurwitz. Hence, the first row of each block is nonnegative. If $A'=S^{-1}J'S$, then $A=A'-\alpha I=S^{-1}(J'-\alpha I)S$. Clearly, $J=J'-\alpha I$ is a direct sum of shifted-companion matrices each of which satisfies the conditions of Theorem 2 so that J is a SNS matrix with negative diagonal entries. □

Because of the above theorem, it can be assumed that the given state-space system in (1) has a sign-unique operating point and is implementable by FOLPFs. A synthesis procedure for implementing such a state-space system is given below:

IV. Synthesis Procedure

Step 1 (Dimensionalization): The variables will be first scaled [5] so that each signal is replaced by a ratio of a signal current to a unit current (which gets cancelled out as the system is linear). The derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ is replaced by $\tau\frac{\mathrm{d}}{\mathrm{d}t}$. Hence, each state-space equation can be written as:

$$\tau\frac{\mathrm{d}I_{x_i}}{\mathrm{d}t}+E_iI_{x_i}=\sum_{j=1}^{n}A'_{ij}I_{x_j}+\sum_{k=1}^{m}B_{ik}I_{u_k}\ i=1,2,\ldots n \quad (5)$$

$$I_{y_i}=\sum_{j=1}^{n}C_{ij}I_{x_j}+\sum_{k=1}^{m}D_{ik}I_{u_k}\quad i=1,2,\ldots p \quad (6)$$

Step 2 (FOLPF implementation): A MITE FOLPF [6], [5] used in the i^{th} first-order equation in (5) is shown in Fig. 1(a). The MITE network satisfies the equation:

$$\frac{CU_T}{\kappa}\frac{\mathrm{d}I_{x_i}}{\mathrm{d}t}+I_{\tau_i}I_{x_i}=I'_{\tau_i}I_{in_i} \quad (7)$$

Fix a value of C and define $I_\tau=\frac{CU_T}{\kappa\tau}$. Define $I_{\tau_i}=E_iI_\tau$. Choose a $\alpha_i>0$ (typically the magnitudes of one of the coefficients in the right hand side of (5)) and define $I'_{\tau_i}=\alpha_iI_\tau$. Hence, the required input current I_{in_i} to the filter is given by $I_{in_i}=\sum_{j=1}^{n}\frac{A'_{ij}}{\alpha_i}I_{x_j}+\sum_{k=1}^{m}\frac{B_{ik}}{\alpha_i}I_{u_k}$.

Step 3 (Multiplier implementation): The multiplications $(\frac{A'_{ij}}{\alpha_i})I_{x_j}$,$(\frac{B_{ik}}{\alpha_i})I_{u_k}$,$C_{ij}I_{x_j}$,$D_{ik}I_{u_k}$ in (5) and (6) are implemented through straightforward methods given in [5]. The inputs I_{u_k} are passed through diode-connected MITEs to generate the voltages V_{u_k} as shown in Fig. 1(b). Hence, I_{x_i} is associated with a voltage V_{x_i} (at the output MITE of the i^{th} FOLPF shown in Fig. 1(a)) and similarly, I_{u_k} is associated with V_{u_k}. The circuits for $(\frac{A'_{ij}}{\alpha_i})I_{x_j}$,$(\frac{B_{ik}}{\alpha_i})I_{u_k}$, shown respectively in Fig. 1(c) and Fig. 1(d), are in terms of these voltages. The products for the output currents are generated in an identical fashion.

Step 4 (Summation): The inputs I_{in_i} to the FOLPFs and the outputs I_{y_i} are found simply by using KCL, through a current mirror if needed as shown in Fig. 1(c) and Fig. 1(d). Also, the output MITE of each FOLPF can be removed unless the state variable I_{x_i} is itself one of the output currents I_{y_i}. Consolidation [5] can be used to remove redundancies whenever possible.

A. Example : SCF filter synthesis

For $\alpha>0$, the SCF state-space equations are implementable by FOLPFs and have a sign-unique operating point. Though the synthesis procedure detailed above can be used directly, a

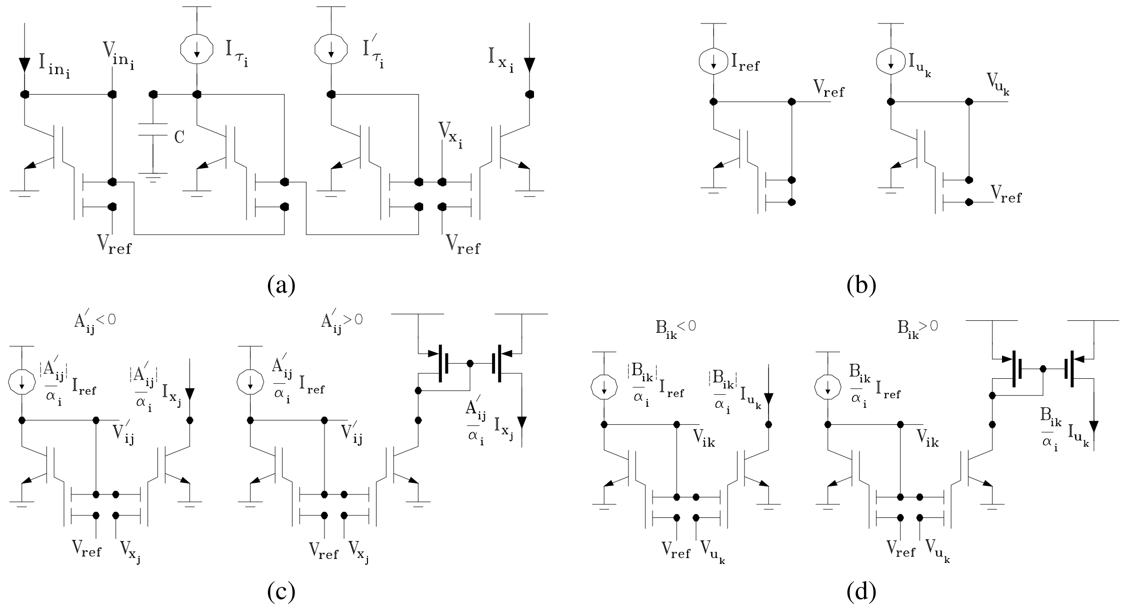

Fig. 1. The circuit blocks used in implementing the first-order equations in (5). (a) The MITE low-pass filter used in the i^{th} equation in (5). (b) The MITE circuits for generating the voltages V_{ref} and V_{u_k} used in the multipliers. (c) The MITE circuits implementing the product $(\frac{A'_{ij}}{\alpha_i})I_{x_j}$ for $A'_{ij} > 0$ and $A'_{ij} < 0$. (d) The MITE circuits implementing the product $(\frac{B_{ik}}{\alpha_i})I_{u_k}$ for $B_{ik} > 0$ and $B_{ik} < 0$.

convenient scaling of the state variables before applying the synthesis procedure results in a much simpler topology. Define $T = diag(1, a_{n-2}, a_{n-3}, \ldots a_0)$. The SCF system in Section III-A is transformed according to $A' = TAT^{-1}$, $B' = TB$, $C' = CT^{-1}$, and $D' = D$. The modified system is given by the following equations:

$$\begin{aligned}
\dot{x}_1 + (\alpha + a_{n-1})x_1 &= u - x_2 - x_3 \cdots - x_n \\
\dot{x}_2 + \alpha x_2 &= a_{n-2}x_1 \\
\dot{x}_3 + \alpha x_3 &= \frac{a_{n-3}}{a_{n-2}}x_2 \\
&\vdots \\
\dot{x}_n + \alpha x_n &= \frac{a_0}{a_1}x_{n-1}
\end{aligned}$$

$$y \;=\; b_{n-1}x_1 + \frac{b_{n-2}}{a_{n-2}}x_2 + \frac{b_{n-3}}{a_{n-3}}x_3 + \cdots \frac{b_0}{a_0}x_n + du$$

It should be noted that the state variable equations are a cascade of FOLPFs with input $u - x_2 - x_3 \cdots - x_n$. Consolidation can be applied to a cascade of FOLPFs since the output MITE of a FOLPF and the input MITE of the FOLPF following it can be removed and the corresponding voltages connected, as shown in Fig. 2. The whole generic SCF filter is shown in Fig. 2. Note that the required multiplier blocks are easily synthesized as described in the synthesis procedure. Also, this block can be used as a "universal active filter" to generate filters of any type and any order. (For those filters that do not pass DC, a offset needs to be applied at the output so that the requirement of positive currents through the MITEs is satisfied.)

V. Uniqueness of the Operating Point

For determining conditions on the synthesized filter such that the operating point is unique, we need a general model for a MITE that covers all regions of operation of the basic translinear element (BJT, MOSFET etc.) constituting the MITE. Based on a model for a MITE that assumes the weighted sum of voltages to be ideal, sufficient conditions on the MITE network topology for the operating point to be unique have been given elsewhere [7]. As only 2-input MITEs are required for this synthesis methodology, even the requirement of ideal weighted summation can be discarded. For the nonideal model of the MITE in Fig. 3, the current through the input gates will be required to zero. The drain current is assumed to be of the form $I = h(V_1, V_2, V_d)$ where $h : (0, V_{dd})^3 \mapsto (0, \infty)$ is a C^1 map satisfying : $\forall\, (V_1, V_2, V_d) \in (0, V_{dd})^3$

$$\begin{aligned}
\text{Transconductance 1} \quad g_{m_1} &\triangleq \frac{\partial h}{\partial V_1} > 0, \\
\text{Transconductance 2} \quad g_{m_2} &\triangleq \frac{\partial h}{\partial V_2} > 0, \qquad (8) \\
\text{Output conductance} \quad g_o &\triangleq \frac{\partial h}{\partial V_d} \geq 0
\end{aligned}$$

In a floating-gate implementation, this is nothing more than the assumption that the (nonideal) weighted summation is monotonically increasing along with the requirements of positive transconductance and nonnegative output conductance of the MOSFET. We can give a similar form for the drain current through a PFET (in a current mirror or a current source) taking care about the signs for the different conductances. A (brief) proof of the uniqueness of the operating point is as follows:

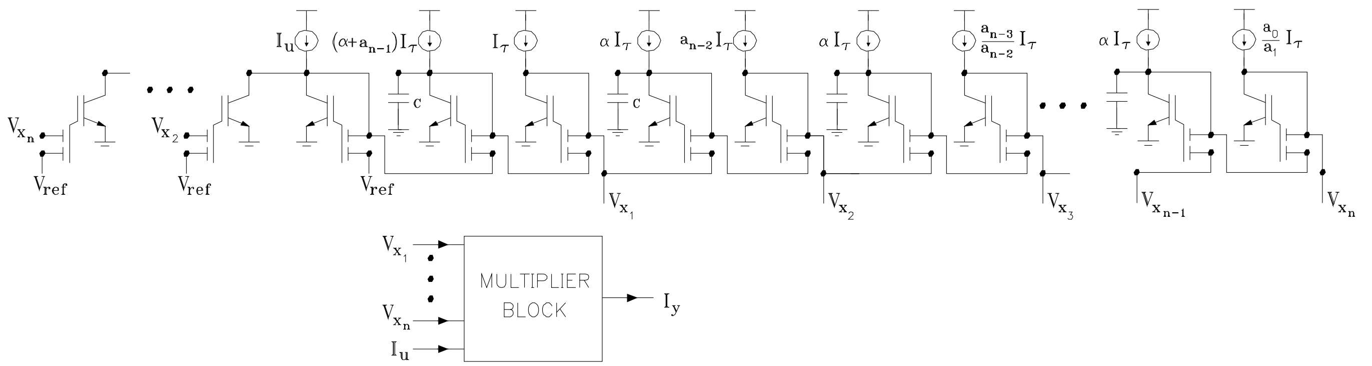

Fig. 2. A generic shifted-companion-form filter. It can be used to generate any linear transfer function. The poles and zeros can be tuned using the bias currents. The multiplier block is implemented as described in Section IV.

Theorem 4: The DC MITE circuit realizing (1) according to the synthesis procedure in Section IV has at most one operating point (with the operating point voltages in $(0, V_{dd})$) if A is a SNS matrix.

Proof : Since all the elements (MITEs, PFETs) are voltage controlled, we can write the nonlinear node equation [8]: $f(V) = 0$ where $f : (0, V_{dd})^k \mapsto \mathbb{R}^k$, V being the vector of drain-to-ground voltages of the MITEs with common drains. It should be noted that the MITEs that are the outputs of products in (6) do not affect the operating point uniqueness. By Theorem 1, it suffices to prove that the Jacobian $Df(V)$ does not change sign patterns for $V \in (0, V_{dd})^n$ and that Df is a SNS matrix. It can be seen from the way the MITEs are connected that $(Df(V))_{ij}$ has a sign independent of V. Df is nothing but the node-admittance matrix of the linear network N obtained by setting the DC sources to zero and replacing the nonlinear elements (PFETs, MITEs) by their small-signal equivalent circuits according to (8). Consider the set $\mathcal{N}$ of linear networks obtained by changing the magnitude alone of different transconductances and conductances in N. Any matrix $M \in \mathcal{Q}(Df)$ is obtained as the node-admittance matrix of a element N' of $\mathcal{N}$. To show that $\det(M) \neq 0$, it suffices to prove that the corresponding network has a unique solution in which the node-to-ground voltages V are zero. We make the following (easily provable) observations about voltages in N' (which correspond to the voltages in Fig. 1):

1) The voltages V_{u_k}, V'_{ij}, V_{ik} and V_{ref} are zero.
2) Wherever the voltage V_{in_i} appears in the (linear) node equations, it can be replaced by $\beta_{ii} V_{x_i}$ where $\beta_{ii} > 0$.
3) If $K = sign(A')$, then for some arbitrary $\beta_{ij} > 0$, $V_{in_i} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_{ij} K_{ij} V_{x_j}$

Combining the results in 2) and 3), it is clear that $LV = 0$ where $V = (V_{x_1}, V_{x_2}, \ldots V_{x_n})^t$ and $L \in \mathbb{R}^{n \times n}$ is given by $L_{ii} = \beta_{ii}$ and for $i \neq j$, $L_{ij} = \beta_{ij} K_{ij}$. By the definition of K, L is in $\mathcal{Q}(A)$. Since A is a SNS matrix, $det(L) \neq 0$ and hence $V = 0$, which implies that all the node-to-ground voltages (in N') are zero. □

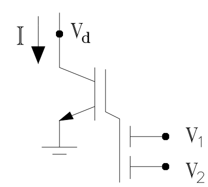

Fig. 3. Symbol for a 2-input Multiple Input Translinear Element (MITE). Ideally, it should obey the law $I = I_s \exp(\frac{\kappa}{U_T}(V_1 + V_2))$.

VI. CONCLUSION

Conditions on the state-space equations for log-domain filters that ensure the uniqueness of the operating point of the resulting circuit have been presented. A synthesis procedure using first-order low-pass filters to implement any log-domain filter has been described. It is proved that the operating point for the synthesized filter is unique. As an example, shifted-companion-form filters of arbitrary type and order are synthesized.

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