# Optimal Synthesis of MITE Translinear Loops

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***Abstract*— A procedure for synthesizing multiple-input translinear element (MITE) networks that implement a given system of translinear–loop equations (STLE) is presented. The minimum number of MITEs required for implementing the STLE, which is equal to the number of current variables in the STLE, is attained. The number of input gates of the MITEs is minimal amongst those MITE networks that satisfy the STLE and have the minimum number of MITEs. The synthesized MITE networks have an unique operating point and, in many cases, the network is guaranteed to be stable in a particular sense. This synthesis procedure exploits the relationship between MITE product–of–power–law (POPL) networks and linear diophantine equations which is explored in detail here.**

## I. Introduction

Translinear circuits are suitable for implementing a wide variety of nonlinear (and linear) systems that can be written as differential equations [1] or as static functions [2]. In *multiple-input translinear element* (MITE) networks [3], the voltage addition in a translinear loop is transformed into voltage addition through a capacitor voltage summer. The $n$–input MITE, as represented in Figure. 1(a), is a $(n+1)$-port circuit element characterized by

$$\begin{aligned} I_i &= 0 \qquad (i = 1, 2, \ldots, n) \\ I_{n+1} &= I_s \exp[\kappa \frac{(w_1V_1 + w_2V_2 + \cdots w_nV_n)}{U_{\mathrm{T}}}], \end{aligned} \tag{1}$$

where $I_i$ and $V_i$ are the port currents and port voltages. $I_s$ is a pre-exponential scaling current and $U_{\mathrm{T}} = kT/q$ is the thermal voltage. $\kappa$ is a dimensionless scaling factor. The $w_i$ are nonnegative weight coefficients, usually integers. $w \triangleq \sum_{i=1}^{n} w_i$ is called the *fan–in* of the MITE [3].

The basic principle of translinear circuits, the Gilbert translinear principle [2], states that, in a closed loop of identical translinear elements comprising an equal number of clockwise and anticlockwise elements,

$$\prod_{n \in CW} I_n = \prod_{n \in ACW} I_n \tag{2}$$

where $CW$ and $ACW$ are the sets of the indices of the elements in clockwise and anticlockwise directions, respectively. Generalizing this relationship, we define a *translinear–loop equation* as a relationship between (positive) variables $I_1, I_2, \ldots I_m$ of the form $\prod_{i=1}^{m} I_i^{a_i} = 1$, where the $a_i$s are integers such that $\sum_{i=1}^{m} a_i = 0$. It should be noted that this definition does not require the presence of actual translinear loops. The synthesis of MITE implementations of translinear–loop equations (or a system thereof), and the equivalent *product–of–power–law* (POPL) equations is discussed in [3]–[6]. This paper describes the synthesis of MITE networks satisfying a given system of translinear–loop equations. The resultant MITE network is optimal with respect to the number of MITEs and the fan–in of the MITES in a certain sense that will be described. This will be followed up with the synthesis subject to constraints in [7]. The theory of linear diophantine equations [8]–[10] is used in the synthesis.

## II. Mathematical Preliminaries

$\mathbb{R}, \mathbb{Q}, \mathbb{Z}$, and $\mathbb{N}$ denote the set of real numbers, rationals, integers, and nonnegative integers, respectively. If $m \le n$, $[m : n]$ is the set $\{m, m+1, \ldots, n\}$. The set of all $m \times n$ matrices whose elements are in $\mathbb{F} \subseteq \mathbb{R}$ is denoted by $\mathcal{M}_{m,n}(\mathbb{F})$. If $A \in \mathcal{M}_{m,n}(\mathbb{F})$, $\alpha \subseteq [1 : m]$, and $\beta \subseteq [1 : n]$, then $A(\alpha, \beta)$ is the matrix formed by the rows and columns of $A$ indexed by $\alpha$ and $\beta$, respectively. $\mathbf{1}_n$ denotes the $n \times 1$ vector with all elements being 1. If $f\colon \mathbb{A} \to \mathbb{B}$, and if $\mathbf{x} = [x_i] \in \mathbb{A}^n$, then $f(\mathbf{x})$ denotes the vector $[f(x_i)] \in \mathbb{B}^n$. If $\mathbf{x} = [x_i]_{i=1}^m$ and $\mathbf{y} = [y_i]_{i=1}^m$, then the notation $\mathbf{x} \gg \mathbf{y}$ means that $x_i \ge y_i$ for all $i \in [1 : m]$ and that there exists a $j \in [1 : m]$ so that $x_j > y_j$.

## III. Translinear Loops

The implementation of a system of translinear loop equations using MITE circuits is now discussed [11]. A *system of translinear–loop equations* (STLE) is defined as a relationship between current variables $I_1, I_2, \ldots, I_m$ of the form

$$\prod_{j=1}^{m} I_j^{a_{ij}} = 1, \quad i \in [1 : l] \tag{3}$$

The matrix $A = [a_{ij}]$ represents the powers to which the currents are raised and will be referred to as the *translinear loop matrix*. Since the powers of interest usually are rational numbers, it follows that without loss of generality, $A \in \mathcal{M}_{l,m}(\mathbb{Z})$ can be assumed. Dimensional consistency requires $A\mathbf{1}_m = 0$. Taking logarithms on both sides of Eq. (3),

$$A \log(\mathbf{I}) = 0 \tag{4}$$

Clearly, it can be assumed that there are no redundant equations in Eq. (3), implying that the rows of $A$ are linearly independent. Hence the following is assumed:

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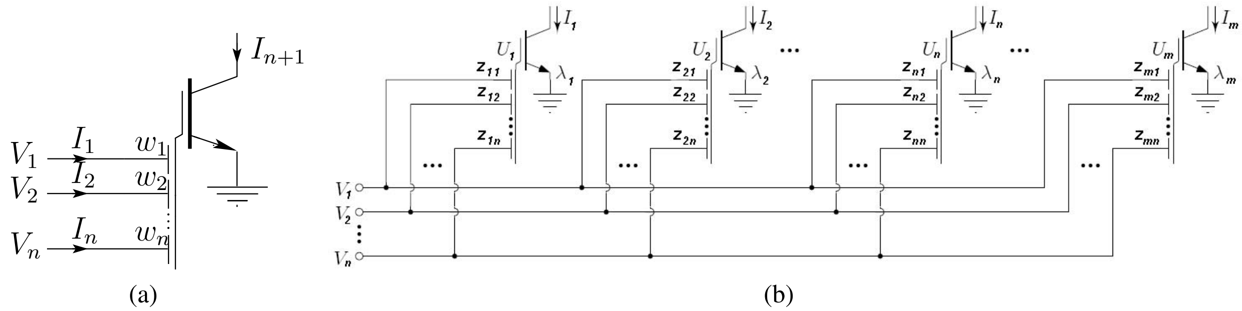


Fig. 1. (a) Symbol for a $n$-input multiple-input translinear element (MITE). (b) The canonical MITE network used to implement STLE Eq. (3). The voltages $V_1, V_2, \ldots, V_n$ are generated by "diode" connecting them to the respective drains of the input MITEs with currents $I_1, I_2, \ldots, I_n$.

*Convention 1:* If $A$ is a translinear loop matrix, then $A$ is full-row-rank; i.e., $\operatorname{rank} A = l$.

## A. Input-Output Separation

Since $\operatorname{rank}(A) = l$, $l$ linearly independent columns of $A$, indexed by $\gamma$, can be chosen. If $\beta = [1 : l]$, then the matrix $A(\beta, \gamma)$ is a nonsingular square matrix. Eq. (4) can thus be written as $A(\beta,\gamma)\log(\mathbf{I}(\gamma)) + A(\beta,\gamma')\log(\mathbf{I}(\gamma')) = 0$, where by definition, the vector $\mathbf{I}(\gamma')$ represents the currents in $\mathbf{I}$ indexed by the indices *not* in $\gamma$. Thus, $\log(\mathbf{I}(\gamma)) = -A(\beta,\gamma)^{-1}A(\beta,\gamma')\log(\mathbf{I}(\gamma'))$. This means that the $n$ currents in $\mathbf{I}(\gamma')$ can be taken to be inputs and the $l$ currents in $\mathbf{I}(\gamma)$ to be outputs to the MITE network. This formulation is nothing but the POPL formulation of [11], in which the output currents are written as products of the input currents raised to different powers. As the criteria ensuring the uniqueness [7] and the stability [4] of the operating point of the POPL network require a separation of the currents into inputs and outputs, the following convention will be followed:

*Convention 2:* The currents in the STLE in Eq. (3) are numbered so that $I_1, I_2, \ldots, I_n$ are the inputs and $I_{n+1}, I_{n+2}, \ldots, I_m$ are the outputs, where $n = m - l$. For this to be valid, $A(\beta,\gamma)$ must be nonsingular, where $\gamma = [n+1 : m]$ and $\beta = [1 : l]$.

# IV. The Synthesis Problem

The objective is to find a suitable connectivity matrix $Z \in \mathcal{M}_{m,n}$, which describes the canonical MITE network in Figure. 1(b), when the translinear loop matrix $A \in \mathcal{M}_{l,m}(\mathbb{Z})$ is given. Hence, it is desired that $\{I_s \exp(\kappa \mathbf{U}/U_{\mathrm{T}}) \mid \mathbf{U} = Z\mathbf{V} \text{ for some } \mathbf{V} \in \mathbb{R}^n\}$, representing the set-theoretic *relation* determined by $Z$, be the same as $\{\mathbf{I} \in \mathbb{R}^m \mid A\log(\mathbf{I}) = 0\}$, which represents the STLE. Consider the vector $\mathbf{U} \in \mathbb{R}^m$ defined by $\mathbf{U} = (U_{\mathrm{T}}/\kappa)\log(\mathbf{I}) - (U_{\mathrm{T}}/\kappa)\log(I_s)\mathbf{1}_m$. Clearly, $\mathbf{I}$ satisfies $A\log(\mathbf{I}) = 0$ iff $A\mathbf{U} = (U_{\mathrm{T}}/\kappa)A\log(\mathbf{I}) - (U_{\mathrm{T}}/\kappa)\log(I_s)A\mathbf{1}_m = 0$. We require $\{\mathbf{U} \mid \mathbf{U} = Z\mathbf{V} \text{ for some } \mathbf{V} \in \mathbb{R}^n\} = \{\mathbf{U} \in \mathbb{R}^m \mid A\mathbf{U} = 0\}$. The former is the range, $\operatorname{Im}(Z)$, of $Z$ and the latter is the kernel, $\ker(A)$, of $A$. Hence, $\operatorname{Im}(Z) = \ker(A)$ is desired. From elementary linear algebra, one can show that $\operatorname{Im}(Z) = \ker(A)$ iff $AZ = 0$ and $\operatorname{rank}(Z) = \operatorname{nullity}(A)$. If Conventions 1 and 2 are taken into account, then the $\operatorname{rank}(Z) = \operatorname{nullity}(A) = n$ requirement can be shown to reduce to $Z$ being of the form $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, where $X \in \mathcal{M}_{n,n}(\mathbb{N})$ is nonsingular. $X$ and $Y$ are called the *input* and *output* connectivity matrices, respectively. Taking into account these constraints as well as the ones in [4], [7], the synthesis problem can be stated as

**Given** $A \in \mathcal{M}_{l,m}(\mathbb{Z})$. If $\gamma = [n+1 : m]$ and $\beta = [1 : l]$, $A(\beta,\gamma)$ is nonsingular.

**Problem** Find a matrix $Z \in \mathcal{M}_{m,n}(\mathbb{N})$ satisfying:

- P1 $AZ = 0$.
- P2 $Z\mathbf{1}_n = w\mathbf{1}_m$ for some $w \in \mathbb{N}$. This ensures that the MITE network is *balanced* [3], [11].
- P3 If $\alpha = [1 : n]$, then $X = Z(\alpha,\alpha)$ is nonsingular. This implies that $\operatorname{rank}(Z) = \operatorname{nullity}(A)$.
- P4 $X$ is a $P_0$ matrix; i.e., all the principal minors of $X$ are nonnegative. This (along with P3) ensures the uniqueness of the operating point [7].
- P5 $X$ is $D$–stable; i.e., the eigenvalues of $DX$ lie in the right–half $s$–plane for all diagonal matrices $D$ with a positive diagonal. This implies that the MITE network is stable in the sense described in [4].

Conditions P4 and P5 assume that voltage $V_i$ is connected to the drain of the MITE with current $I_i$, for $i \in [1 : n]$. Matching issues can be taken care of by programming the floating-gates.

Some of the important parameters that need to be minimized are the number of MITEs and the fan–in of each MITE. Increasing either of these parameters usually results in a increase in area. For the same floating–gate capacitance value, if the fan–in is increased, the maximum frequency of operation of the circuit decreases. The synthesis methods in [3]–[5] are mainly for implementing each equation in the STLE separately. Consolidation is then used to remove redundant MITEs. If full consolidation is not possible, then the network has copies of the input currents and hence the procedure does not minimize the number of MITEs. On the other hand, these methods can potentially reduce the fan–in, and it follows from [3], [4] that it can be reduced to the minimum possible value of 2. However, there is no procedure to minimize the number of MITEs once the fan–in is fixed at some value.

The procedure presented here solves the STLE as a whole rather than solving each equation separately. Further,

1) The minimum number of MITEs required for implementing Eq. (3), viz. $m$, is attained.

2) The minimum fan–in $w_{\min}$ is obtained amongst all MITE networks *with $m$ MITEs* implementing the STLE.

## V. Operating Point Uniqueness and Stability

Condition P4 is not robust in the sense that a small perturbation of the elements can lead to some principal minors of $X$ (the zero ones) to become nonnegative. This can be avoided by requiring $X$ to be a $P$–matrix [12], in which all the principal minors are positive (not just nonnegative). Hence, for synthesis purposes, instead of P4, the condition P4´ below shall be used:

P4´ $X$ is a $P$–matrix; i.e., all principal minors of $X$ are positive.

An algorithm of order $O(2^n)$ for testing P4´has been proposed [13] which is used in the synthesis procedure. The condition P5; i.e., $X$ is $D$–stable is actually a sufficient condition for P4 (and P3) but does not imply that $X$ is a $P$–matrix. However, no finitely verifiable necessary and sufficient condition exists for checking $D$–stability [12], though there are useful sufficient conditions [14]. A synthesis procedure for POPL networks, which is not necessarily minimal, but which satisfies P1-P5 has also been proposed [6]. This method is applicable to STLEs and can be used to get an estimate for $w_{\min}$.

## VI. Solution Methodology

The solution(s) of the synthesis problem taking into account conditions P1, P2, and P3 is first discussed.

If $Z$ is written in terms of its columns; i.e., $Z = [\mathbf{z}_1\ \mathbf{z}_2 \ldots \mathbf{z}_n]$ and noting that $AZ = [A\mathbf{z}_1\ A\mathbf{z}_2 \ldots A\mathbf{z}_n]$, the problem (P1, P2, and P3) is equivalent to finding a set $\{\mathbf{z}_i\}_{i=1}^n$ with $\mathbf{z}_i \in \mathbb{N}^m$ so that the following are satisfied:

R1 $A\mathbf{z}_i = 0 \qquad i \in [1:n]$

R2 $\Sigma_{i=1}^n \mathbf{z}_i = w\mathbf{1}_m$

R3 The vectors $\{\mathbf{z}_i\}$ are linearly independent. This is equivalent to the vectors $\{\mathbf{x}_i\}$ being linearly independent, where $\mathbf{x}_i = \mathbf{z}_i([1:n])$.

*Theorem 1 (Completion Theorem):* If $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n-1}$ with $\mathbf{z}_i \in \mathbb{N}^m$ are such that

A1 $A\mathbf{z}_i = 0 \qquad i \in [1:n-1]$

A2 The vectors $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n-1}, \mathbf{1}_m\}$ are linearly independent,

then $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ satisfies R1, R2, and R3 with $w = \|\sum_{i=1}^{n-1} \mathbf{z}_i\|_\infty \triangleq \max \sum_{i=1}^{n-1} \mathbf{z}_i$, where $\mathbf{z}_n = w\mathbf{1}_m - \sum_{i=1}^{n-1} \mathbf{z}_i$.

*Proof :* Let $S = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n\}$. $\mathbf{z}_n \in \mathbb{N}^m$ by the definition of $w$. Clearly, $A\mathbf{z}_n = wA\mathbf{1}_m - \sum_{i=1}^{n-1} A\mathbf{z}_i = 0$. Hence $S$ satisfies R1. R2 is valid by the definition of $\mathbf{z}_n$. To check R3, let $\sum_{i=1}^n \alpha_i \mathbf{z}_i = 0$. Using the definition of $\mathbf{z}_n$, this is equivalent to $\sum_{i=1}^{n-1}(\alpha_i - \alpha_n)\mathbf{z}_i + \alpha_n w \mathbf{1}_m = 0$. By A2, we have $\alpha_n w = 0$ and $\alpha_i - \alpha_n = 0$ for $i \in [1:n-1]$. Since $w \neq 0$, we have $\alpha_n = 0$, which implies $\alpha_i = 0$. Hence R3 is also satisfied by $S$. □

Hence the problem of satisfying R1, R2, and R3 reduces to finding $\mathbf{z}_1, \mathbf{z}_2, \ldots \mathbf{z}_{n-1}$ satisfying A1 and A2. Since we are interested in minimizing the fan–in $w$, it seems intuitively obvious that it suffices to "minimize" the $\mathbf{z}_i$ in some sense. This notion is made precise in the following.

### A. Linear Diophantine Equations

Let us consider the *linear Diophantine equation*

**Given** $A \in \mathcal{M}_{l,m}(\mathbb{N})$

**Problem** Find $\mathcal{S} = \{\mathbf{z} \in \mathbb{N}^m |\ A\mathbf{z} = 0\}$

There exists a finite set $\mathcal{H} \subset \mathcal{S}$, called the *Hilbert basis* or the set of *minimal solutions* of the diophantine equation, such that every element of $\mathcal{S}$ can be written as a nonnegative integral combination of the elements of $\mathcal{H}$ [8], [9]. The elements of $\mathcal{H}$ are minimal in the sense that if $\mathbf{u} \in \mathcal{H}$, then there is no other $\mathbf{v} \in \mathcal{S},\ \mathbf{v} \neq 0$ such that $\mathbf{u} \gg \mathbf{v}$.

Various algorithms exist for finding the set of minimal solutions [10]. The algorithm used here is the so-called ABCD algorithm [8], [9]. A brief description follows.

Starting with the standard basis for $\mathbb{N}^m$, if at some stage $\mathbf{x} = [x_i] \in \mathbb{N}^m$ is not a solution,

C1 If $A\mathbf{x}.A\mathbf{e}_j < 0$, then increment $x_j$ by 1.

If after incrementing, $\mathbf{x}$ is greater than ( i.e., $\gg$) any previous solution, then it is removed. Of course, if $A\mathbf{x} = 0$, it is added to the minimal solution set. The process stops after a finite number of steps and all and only the minimal solutions are found. The actual algorithm used here is a more efficient refinement of the above idea [8], [9].

### B. Existence and Construction of Solution

The following theorem shows that minimal fan–in POPL MITE networks can be constructed using the vectors in the minimal solution set $\mathcal{H}$.

*Theorem 2 (Construction Theorem):*

1) There exist vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n-1}$ with $\mathbf{z}_i \in \mathcal{H}$ satisfying A1 and A2.
2) The minimum possible fan–in is also obtained as $w_{\min} = \min\{\|\sum_{i=1}^{n-1} \mathbf{z}_i\|_\infty |\ \mathbf{z}_i \in \mathcal{H}\}$; i.e., the fan–in can be minimized by appropriately choosing elements of $\mathcal{H}$, which is a finite set compared to the solution set $\mathcal{S} = \{\mathbf{z} \in \mathbb{N}^m |\ A\mathbf{z} = 0\}$, which is infinite.

*Proof of 1:* By Convention 1, $\operatorname{rank}(A) = l$, hence $\operatorname{nullity}(A) = m - l = n$. Since $A \in \mathcal{M}_{l,m}(\mathbb{Q})$, it can be considered as a linear transformation from $\mathbb{Q}^m$ onto $\mathbb{Q}^l$. Hence, $\ker(A) = \{\mathbf{z} \in \mathbb{Q}^m |\ A\mathbf{z} = 0\}$, has dimension $n$. Since $A\mathbf{1}_m = 0$, a basis for $\{\mathbf{z} \in \mathbb{Q}^m |\ A\mathbf{z} = 0\}$ can be constructed by suitably appending $n-1$ more vectors $\mathbf{z}'_1, \mathbf{z}'_2, \ldots, \mathbf{z}'_{n-1}$. It is clear that by multiplying all the $\mathbf{z}'_i$s by suitable integers, we can assume $\mathbf{z}'_i \in \mathbb{Z}^m$. Let $-c_i$ be the most negative integer amongst the components of $\mathbf{z}'_i$. The set $\{\mathbf{z}'_1 + c_1\mathbf{1}_m, \mathbf{z}'_2 + c_2\mathbf{1}_m, \ldots, \mathbf{z}'_{n-1} + c_{n-1}\mathbf{1}_m, \mathbf{1}_m\}$ is clearly a subset of $\mathbb{N}^m$ and can be easily shown to be linearly independent. Since $A(\mathbf{z}'_i + c_i\mathbf{1}_m) = A\mathbf{z}'_i + c_i A\mathbf{1}_m = 0$, it has been shown that we can choose $\{\mathbf{z}_i\}_{i=1}^{n-1} \subset \mathbb{N}^m$ satisfying A1 and A2.

By the definition of $\mathcal{H}$, each $\mathbf{z}_i$ constructed above can be written as a nonnegative linear combination of elements $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k$ of $\mathcal{H}$. Hence $\mathbf{z}_i = \sum_{j=1}^k \alpha_{ij}\mathbf{v}_j$, where $\alpha_{ij} \in \mathbb{N}$. Let $\mathbf{x}_i = \mathbf{z}_i([1:n])$ and $\mathbf{u}_i = \mathbf{v}_i([1:n])$. By R3, $\det[\mathbf{x}_1\, \mathbf{x}_2 \cdots \mathbf{x}_{n-1}\, \mathbf{1}_n] \neq 0$. Since the determinant is a linear function of each of the column vectors, $\det[\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_{n-1} \mathbf{1}_n]$

can be written as a linear combination of determinants of the form $\det[\mathbf{u}_{i_1}\ \mathbf{u}_{i_2}\ \cdots\ \mathbf{u}_{i_{n-1}}\ \mathbf{1}_n]$, where $i_1, i_2, \ldots, i_{n-1}$ are integers between 1 and $k$. We cannot have all these determinants to be zero, else $\det[\mathbf{x}_1\ \mathbf{x}_2\ \cdots \mathbf{x}_{n-1}\ \mathbf{1}_n] = 0$. Hence there exist vectors $\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_{n-1}}$ in $\mathcal{H}$ satisfying A1 and A2. This proves part 1.

*Proof of 2:* Let $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n-1}\} \subset \mathbb{N}^m$ satisfying A1 and A2 have the minimum fan–in; i.e., $\|\sum_{i=1}^{n-1} \mathbf{z}_i\|_\infty = w_{\text{min}}$. If $\{\mathbf{z}_i\}_{i=1}^{n-1}$ is not a subset of $\mathcal{H}$, then since A1 is satisfied, each $\mathbf{z}_i = \sum_{j=1}^{k} \alpha_{ij}\mathbf{v}_j$, where $\alpha_{ij} \in \mathbb{N}$. Proceeding as in the previous part, it can be shown that for some $i_1, i_2, \ldots, i_{n-1}$, the vectors $\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \ldots, \mathbf{v}_{i_{n-1}}$ satisfy A1 and A2. However, since $\mathbf{v}_{i_j}$ is part of the nonnegative linear expansion of $\mathbf{z}_j$, it must be true that $\alpha_{ji_j} > 0$. Hence, $\mathbf{z}_j \gg \mathbf{v}_{i_j}$, which implies that $\sum_{j=1}^{n-1} \mathbf{z}_j \gg \sum_{j=1}^{n-1} \mathbf{v}_{i_j}$. Since, each of the vectors involved are nonnegative, it is clear that $w_{\text{min}} \geq \|\sum_{j=1}^{n-1} \mathbf{v}_{i_j}\|_\infty$. By the definition of $w_{\text{min}}$, $\|\sum_{j=1}^{n-1} \mathbf{v}_{i_j}\|_\infty = w_{\text{min}}$. $\square$

The above theorem provides a method to generate MITE networks with minimum fan–in.

## VII. Synthesis Algorithm

**Given** $A \in \mathcal{M}_{l,m}(\mathbb{Z})$. $A(\beta, \gamma)$ is nonsingular.

Initialize the fan–in value $w$ by using the fan–in obtained from the algorithm in [6]. Let the set of minimal connectivity matrices $\mathcal{V} := \emptyset$, initially.

1) Find $\mathcal{H}$, the *finite* set of minimal solutions of $A\mathbf{z} = 0$ using the ABCD algorithm [8], [9].
2) Choose $S' := \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{n-1}\} \subset \mathcal{H}$.
3) Find the fan–in $w' := \|\sum_{i=1}^{n-1} \mathbf{z}_i\|_\infty$. If $w' > w$, go to Step 2.
4) Check if $S'$ satisfies A2. If no, go to Step 2 else use Theorem 1 to find $S := \{\mathbf{z}_i\}_{i=1}^{n}$ satisfying R1, R2 and R3.
5) Check if a permutation $\sigma$ of $[1:n]$ exists such that the matrix $Z := [\mathbf{z}_{\sigma(1)}\ \mathbf{z}_{\sigma(2)} \cdots \mathbf{z}_{\sigma(n)}]$ satisfies P4´. If not, go to Step 2. If yes, let $\mathcal{B}$ be the set of such $Z$ matrices satisfying P4´.
6) If $w' = w$, then $\mathcal{V} := \mathcal{V} \bigcup \mathcal{B}$. If $w' < w$, then $\mathcal{V} := \mathcal{B}$.
7) If all possibilities of $S'$ in Step 2 are not exhausted, repeat the sequence from Step 2.
8) Check if $X = Z([1:n], [1:n])$ satisfies the sufficiency and necessary conditions for $D$–stability [12], [14] for all $Z \in \mathcal{V}$. If $X$ is shown to be not $D$–stable, $\mathcal{V} := \mathcal{V} \setminus \{Z\}$.

## VIII. Example

The STLE is $I_1 I_2^{-2} I_3^2 I_6^{-1} = 1$; $I_1 I_2^{-2} I_3 I_5 I_7^{-1} = 1$; $I_1 I_2^{-2} I_4^2 I_8^{-1} = 1$, which is required in the construction of a rms-to-dc converter [5]. Here

$$A = \begin{bmatrix} 1 & -2 & 2 & 0 & 0 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -2 & 0 & 2 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The minimal solutions set $\mathcal{H}$ (written as a matrix) and corresponding connectivity matrix $Z$ are found to be

$$\mathcal{H} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \quad Z = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

Clearly, $w_{\text{min}} = 2$. It can be verified that this $Z$ satisfies P1-P5. Synthesis by means of other methods [5], [6] gives a non-minimal fan–in of 3.

## IX. Conclusion

A new synthesis procedure for implementing systems of translinear–loop equations using MITEs is presented. This procedure results in minimal number of MITEs and minimal fan–in (for the minimum number of MITEs). The relationship between minimal fan–in of MITE networks and minimal solutions of linear Diophantine equations is shown. The resulting MITE networks have a unique operating point and their unconditional stability is tested with available methods.


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