2-MITE Product–of–Power–Law Networks

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***Abstract*— A 2-MITE is a multiple-input translinear element with two input gates. In this paper, different properties of networks of 2-MITEs are derived, especially in the case of product–of–power–law (POPL) networks, in which the output currents are products of the inputs raised to different powers. It is found that conditions ensuring the uniqueness and stability of the operating point in 2-MITE networks are less stringent than those for MITE networks with higher number of input gates. This simplifies the synthesis of these networks considerably. A graph-theoretic approach to the analysis of 2-MITE networks is presented. Necessary conditions for a set of power-law equations to be implementable by 2-MITE networks are derived. Sufficient conditions for the same are presented for the case of POPL networks with one output.**

I. Introduction

A multiple-input translinear element (MITE) is a generalization of the basic translinear element (BJT in forward-active region or a MOSFET in subthreshold region) to the case of multiple input voltages; the output current I_{d} in a MITE is an exponential of a weighted sum of input voltages. Mathematically,

$$I_{\mathrm{d}} = I_s \exp\left[\kappa \frac{(w_1 V_1 + w_2 V_2 + \cdots w_n V_n)}{U_{\mathrm{T}}}\right], \tag{1}$$

I_s is a scaling current, κ is a dimensionless coefficient and both will be taken to be the same for all MITEs under ideal conditions. U_{T} is the thermal voltage. The w_i's are nonnegative integer weight coefficients. $w \triangleq \sum_{i=1}^{n} w_i$ is called the *fan–in* of the MITE [1]. This paper is concerned with *2-MITEs* which are defined as MITEs with fan–in 2. Applications of MITE networks are discussed in [2]–[4].

Any translinear circuit, at the fundamental level, requires the synthesis of translinear loops. Mathematically speaking, the synthesis of the following set of equations is required:

$$I_{\mathrm{o}i} = \prod_{j=1}^{n} I_j^{\Lambda_{ij}}, \quad i = 1, 2, \ldots, m \tag{2}$$

where $\sum_{j=1}^{m} \Lambda_{ij} = 1$. A standard circuit called the product–of–power–law (POPL) network, shown in Fig. 1(a), is used to implement these kinds of equations [1]. Two features of this network contributing to its size are the *number of MITEs* and the *number of input gates (the fan–in) of a MITE*. The requirement of κ being the same for all MITEs translates to all the MITEs in a MITE network having the same fan–in [5]. Synthesis procedures that aim at reducing the number of MITEs are described in [6], [7]. This paper, in contrast, concentrates on MITE networks with the minimum possible fan–in, namely 2. MITE circuits designed using the ideal expressions do not always have unique or stable operating points [8], [9]. These properties are shown to be automatically satisfied for 2-MITE POPL networks (under some mild assumptions) in Section III. 2-MITE POPL networks are then analyzed using a graph-theoretic formulation and shown to belong to a particular class of digraphs in Section IV. Necessary conditions for a power-matrix Λ to be implementable as a 2-MITE POPL network are then developed in Section V. They are extended to sufficient conditions in the case of a POPL MITE network with a single output in Section VI.

II. Mathematical Preliminaries

The terminology for directed graphs (digraphs) used here follows [10]. A 1-factor of a digraph G is a spanning subgraph of G which is regular of degree 1 (i.e., both in-degree and out-degree is 1 for all vertices). A 1-factorial connection from i to j of a digraph G is a spanning subgraph G which contains a directed path P from i to j and a set of vertex-disjoint directed circuits that include all the vertices of G other than those in P. The weight $w(H)$ of a subgraph H of a weighted digraph G is the product of weights of edges in H. $\mathbf{1}_n$ denotes the $n \times 1$ vector with all elements being 1.

III. Uniqueness and stability of Operating Point

In this section, we show that 2-MITE POPL networks have a unique and stable operating point under the assumption that its input connectivity matrix has a positive diagonal.

A POPL network is determined by the input-connectivity matrix $X = [x_{ij}]$ and the output connectivity matrix $Y = [y_{ij}]$, as shown in Fig. 1(a). An input-output relationship of the form given by Eq. (2) with $\Lambda = YX^{-1}$ results when X is nonsingular. In particular, a 2-MITE POPL network also satisfies

1) $X\mathbf{1}_n = 2\mathbf{1}_n$ and $Y\mathbf{1}_n = 2\mathbf{1}_m$ (as the fan–in is two).
2) $x_{ij}, y_{ij} \in \{0, 1, 2\}$ (since x_{ij} and y_{ij} are nonnegative integers).

The synthesis problem is the reverse process, that of finding suitable X and Y given Λ. We will say that $\Lambda_{m\times n}$ (or Eq. (2)) is *2-MITEable* if a 2-MITE POPL network satisfies Eq. (2) *without using any copies* of the input currents i.e., the number of MITEs is $m + n$.

Ideally, the necessary and sufficient condition for the circuit in Fig. 1(a) to have an unique operating point is "$\det(X) \neq 0$". The multiple feedback loops present in MITE circuits can,

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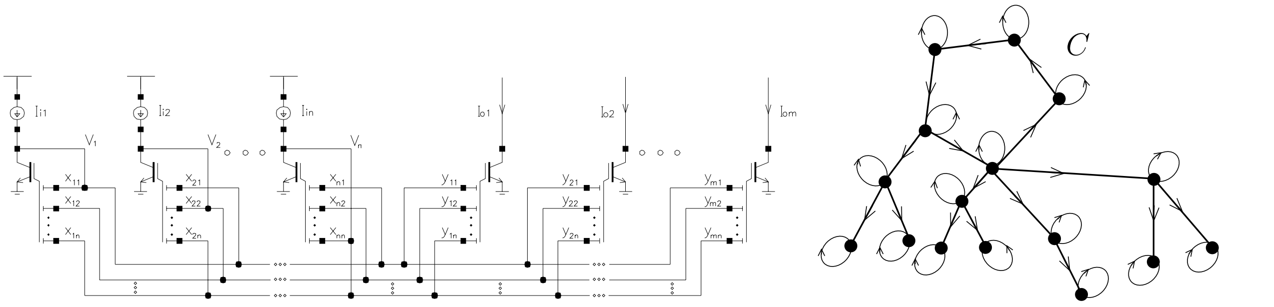

(a) (b)

Fig. 1. (a) The general form of the MITE network implementing a POPL function. The output currents are a product of the input currents raised to different powers. (b) A component of the Coates graph (with directed circuit C) $G_c(X)$ of the input-connectivity matrix X of a 2-MITE POPL network.

however, cause multiple operating points [8]. The following condition suffices to ensure that the operating point is unique:

P1 X is nonsingular and is a P_0 matrix (i.e., X has nonnegative principal minors)

This implies, in particular, that $x_{ii} \geq 0$. We will make the following stronger assumption:

Assumption 1: The input connectivity matrix X of a POPL network has a positive diagonal.

A POPL MITE network described by input-connectivity matrix X is stable (in the sense of [9]) if:

P2 X is D-stable. i.e., DX must be positive-stable for all diagonal matrices D with positive diagonal.

X satisfies **P**1 if it satisfies **P**2 [11]; however, there is no known finitely testable characterization for D-stability for matrices of order greater than three.

For a 2-MITE POPL network, $x_{ii} (> 0)$ is either 1 or 2. Since the rows of X sum to 2, we can write $X = I_n + X'$, where X' has exactly one nonzero entry, namely 1, in each row. Hence, for every row k, we can define a $\alpha(k)$ such that $[X']_{k\alpha(k)}$ is one. Clearly, X is *(row) diagonally dominant*, though not necessarily *strictly row diagonally dominant*. The following theorem then implies that X is D–stable.

Theorem 1: If $A = [a_{ij}] \in \mathcal{M}_n$ is nonsingular, row-diagonally dominant, and has a positive diagonal, then A is D–stable (and hence is also nonsingular and a P_0 matrix).

Proof : Consider $DA = [d_i a_{ij}]$ where the diagonal matrix D has $d_{ii} > 0$. Geršgorin's theorem [11] tells us that the eigenvalues of DA lie in the union of n discs

$$G(DA) = \bigcup_{i=1}^{n} \{\lambda \in \mathbb{C} \, : \, |\lambda - d_i a_{ii}| \leq \sum_{j \neq i}^{n} |d_i a_{ij}|\} \quad (3)$$

The conditions $d_i a_{ii} > 0$ and $d_i a_{ii} \geq \sum_{j \neq i}^{n} |d_i a_{ij}|$ imply that each of the discs lies in the open right half s-plane with the possible exception of including 0. However, the case $\lambda(DA) = 0$ would imply that $\det(A) = 0$, which has been excluded by hypothesis. □

Though not shown here, X is diagonally stable (i.e., has a positive diagonal Lyapunov solution) and hence the D–stability is maintained under small perturbations of the elements of X [12], even though the row-diagonal dominance is not preserved.

IV. 2-MITE NETWORK GRAPHS

In this section, the structure of the Coates graphs of the input connectivity matrices of 2-MITE POPL networks is analyzed.

Restricting both the number of MITEs and the fan–in of a MITE also restricts the possible power matrices that are obtainable from a POPL network. If the fan–in is fixed at 2, it is necessary to find out which powers are obtainable before increasing the number of MITEs suitably. To this end, we take a graph theoretic approach to determine $\Lambda = YX^{-1}$ for a 2-MITE network. To find X^{-1}, we use the method of Coates graphs [10]. Every $A = [a_{ij}] \in \mathcal{M}_n$ corresponds to a weighted digraph $G_c(A)$ with vertices $\{1, 2, \ldots, n\}$ such that there is directed edge (j, i) with weight a_{ij} if $a_{ij} \neq 0$.

Theorem 2: The input-connectivity matrix X of a 2-MITE POPL network can be written as $X = I_n + X'$. Then,

1) Each component G of $G_c(X')$ has a unique directed circuit (self-loops are directed circuits of length 1).
2) If the directed circuit in the digraph G is contracted to a single vertex v, then the resulting digraph is a rooted tree [13] with v as the root.

Proof of 1*:* Every vertex i in $G_c(X')$ has in-degree 1, and if $(j, i) \in E$, then $j = \alpha(i)$. Hence, we can define the *parent* $\alpha(i)$ and a sequence of *ancestors* $\{\alpha^k(i)\}$ for every vertex i. If vertex $j \neq i$ is an ancestor of vertex i, then we say $j \prec i$. We define $j \preceq i$ to mean that either $j \prec i$ or $j = i$. For any vertex i, consider the sequence $\{i, \alpha(i), \alpha(\alpha(i)), \ldots\}$. Since there are only n vertices, the sequence cannot have distinct elements, and hence there exists a vertex j and an integer $p > 0$ such that $\alpha^p(j) = j$. This corresponds to a directed circuit of length *at most* p in $G_c(X')$ and implies that the sequence of ancestors of any vertex i eventually leads to a directed circuit C. We will say that i is *descended* from C.

The relation of being descendants of the same directed circuit is clearly an equivalence relation. We will show that the equivalence classes are the vertex sets of components of $G_c(X')$. If not, there is a (undirected) path P beginning from an equivalence class and ending in a different equivalence class. It is clear that there is an edge $(j, i) \in P$ where the vertices j and i belong to different equivalence classes. However, this implies that $j \prec i$ and hence must be descended from the same directed circuit as i, which contradicts the definition of the classes. Each equivalence class being obviously connected, it follows that each is a component of $G_c(X')$.

Proof of 2: If the directed circuit in a component G is contracted into a single vertex v, it follows that the sequence of ancestors of any vertex i in G (that was not in the directed circuit) now ends at v. Hence, there is a directed path from v to each vertex in G. That this directed path is unique follows from the fact that each vertex has a unique parent. By a characterization of rooted trees [13], G is a rooted tree. □

This characterization of 2-MITE POPL networks enables us to find simple expressions for X^{-1}, as given below.

V. Necessary conditions

Using the method of Coates graph [10], we now derive expressions for X^{-1} and $\Lambda = YX^{-1}$. The determinant of $X \in \mathcal{M}_n$ is given by

$$\det(X) = \sum_H (-1)^{n-L_H} w(H) \tag{4}$$

where H is a 1-factor of $G_c(X)$, and L_H is the number of directed circuits in H. The cofactor Δ_{ij} of x_{ij} is given by

$$\begin{aligned} \Delta_{ii} &= \sum_H (-1)^{n-1-L_H} w(H) \\ \Delta_{ij} &= \sum_{H_{ij}} (-1)^{n-1-L'_H} w(H_{ij}), \quad i \neq j \end{aligned} \tag{5}$$

where H is a 1-factor in the graph obtained by removing i from $G_c(X)$, H_{ij} is a 1-factorial connection in $G_c(X)$ from vertex i to vertex j, and L_H and L'_H are the numbers of directed circuits in H and H_{ij}, respectively.

If $G_c(X)$ is not connected, then by reordering the rows and columns of X, we can write X as a direct sum of matrices X_i that are connected (representing the components of $G_c(X)$). Since X^{-1} is the direct sum of the individual inverses, for finding X^{-1}, it suffices to assume that $G_c(X)$ is connected.

Some definitions are in order:

Definition 1: When n is a nonnegative integer, we define $(-)^n$ to be $(-1)^n$. $(-)^\infty$ is defined to be 0.

Definition 2: The distance $d(i,j)$ is defined as the length of the shortest directed path *from* vertex j *to* vertex i, if a directed path exists. If no directed path exists from j to i, then $d(i,j)$ is defined to be ∞. $d(i,i)$ is 1 if vertex i is attached to a self-loop and is defined to be 0 otherwise.

We will avoid the details of the calculations involved in finding the inverse using the formulae in Eq. (4) and Eq. (5) due to lack of space. If $G_c(X)$ is connected, and C is the unique directed circuit in $G_c(X')$ with k edges in it, then

$$\det(X) = 1 + (-1)^{k+1} \tag{6}$$

Clearly, X is nonsingular iff k is odd in which case $\det(X) = 2$. X^{-1} is then given by

$$[X^{-1}]_{ij} = \frac{[\mathrm{adj}(X)]_{ij}}{\det(X)} = \begin{cases} (-)^{d(i,j)} & \text{if } j \text{ is not in } C \\ \frac{1}{2}(-)^{d(i,j)} & \text{if } j \text{ is in } C. \end{cases} \tag{7}$$

To find $\Lambda = YX^{-1}$, since $Y\mathbf{1}_n = 2\mathbf{1}_m$, it follows that every row in Y contains either a 1 in two different columns or a 2 in a single column; the other elements in the row being 0. Hence, we can write $y_{ij} = \delta_{j\beta(i)} + \delta_{j\gamma(i)}$, where δ is the Kronecker delta function. From this, we have

$$\Lambda_{ij} = \begin{cases} (-)^{d(\beta(i),j)} + (-)^{d(\gamma(i),j)} & \text{if } j \text{ is not in } C \\ \frac{1}{2}((-)^{d(\beta(i),j)} + (-)^{d(\gamma(i),j)}) & \text{if } j \text{ is in } C. \end{cases} \tag{8}$$

The only possible values for $(-)^{d(i,j)}$ are $-1, +1$ and 0. Hence it follows that

$$\begin{aligned} &\Lambda_{ij} \in \{-2,-1,0,1,2\}, \text{ if } j \text{ is not in } C \\ &\Lambda_{ij} \in \{-1,-\frac{1}{2},0,\frac{1}{2},1\}, \text{ if } j \text{ is in } C \end{aligned} \tag{9}$$

Thus, it is clear that the same column in Λ cannot have both a ± 2 and a $\pm 1/2$. For the i^{th} row, if Λ_{ij} is $\pm 1/2$, then it means that j is in C and that there is a directed path from j to either $\beta(i)$ or $\gamma(i)$ but not both. From Theorem 2, it follows that $\beta(i)$ and $\gamma(i)$ belong to two different components. From this, it is clear that Λ_{ij} can not be ± 2 for any j. Similarly, it can be shown that if $\Lambda_{ij} = \pm 2$, then no element in the same row can be $\pm 1/2$. In summary, the following necessary conditions can be concluded:

Condition 1: A power matrix Λ is 2-MITEable *only if*

1) $\Lambda_{ij} \in \{-2,-1,-1/2,0,1/2,1,2\}$
2) No row or column of Λ can contain both a ± 2 and a $\pm 1/2$.

Example: Neither $\Lambda = [1\ 1\ .5\ .5\ -2]$ nor $\Lambda = \left(\begin{smallmatrix} 1 & -1 & .5 & .5 \\ -2 & 1 & 2 & 0 \end{smallmatrix}\right)$ is 2-MITEable.

Also, any leaf j (vertex of out-degree 0) in $G_c(X')$ should be a $\beta(i)$ or a $\gamma(i)$, for otherwise the entire j^{th} column in Λ will be 0, which renders I_j in Eq. (2) useless.

VI. Single-Output POPL Networks

We discuss the conditions for a Λ matrix to be 2-MITEable for the single-output case i.e., Λ is a row vector. Let $y_j = \delta_{j\beta} + \delta_{j\gamma}$. Without loss of generality, we can assume that Λ_j is nonzero for any j, since otherwise the input I_j becomes redundant in Eq. (2).

Case 1: β and γ are in the same component, with associated direct circuit C.

The only possible Coates graph is of the form shown in Fig. 2(a) (All other cases can be rejected because Λ_j becomes zero for some j). Different subcases that can be considered are $\gamma = \delta$ (Fig. 2(c)), $\delta = \gamma = \beta$ (Fig. 2(b)), $\delta = \gamma = \beta = \epsilon$ (Fig. 2(d)). Only the general case is considered here; the powers in the other cases behave as shown in the figures. Since $\Lambda_\delta \neq 0$, $(-1)^{d(\beta,\delta)} + (-1)^{d(\gamma,\delta)} \neq 0$. This means that $d(\beta,\delta) - d(\gamma,\delta)$ must be even. Using Eq. (8), we get:

$$\Lambda_j = \begin{cases} (-1)^{d(\gamma,j)} & \text{if } \delta \prec j \preceq \gamma \\ (-1)^{d(\beta,j)} & \text{if } \delta \prec j \preceq \beta \\ 2(-1)^{d(\beta,j)} & \text{if } \epsilon \prec j \preceq \delta \\ (-1)^{d(\beta,j)} & \text{if } j \in C \end{cases} \tag{10}$$

Example: This shows that $\Lambda = [2\quad 2\quad -1\quad -1\quad -1]$ is not 2-MITEable but $\Lambda = [1\quad -1\quad 2\quad -2\quad 2\quad -1]$ is.

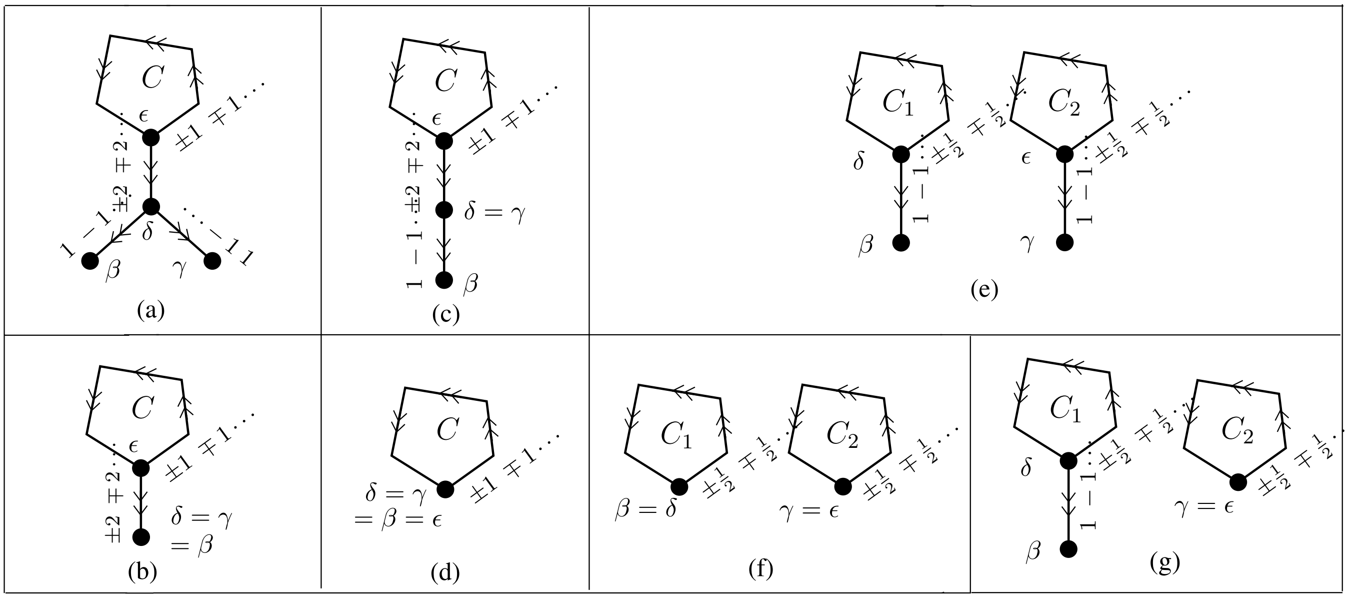

Fig. 2. $G_c(X')$ for 2-MITE POPL networks with single outputs. (a) and (e) are the only possible general cases. (c),(b),(d) are the subcases of (a) while (f) and (g) are the subcases of (e). The sequence of Λ_j values are shown for each section. The double arrows indicate a sequence of directed edges forming a directed path.

Case 2: β and γ are in different components, with associated directed circuits C_1 and C_2.
The only possibility is as shown in Fig. 2(e), allowing for $\beta = \delta$ or $\gamma = \epsilon$ (The subcases with both $\beta = \delta$ and/or $\gamma = \epsilon$ are shown in Fig. 2(f) and (g)). It is a easy consequence of Eq. (8) that

$$\Lambda_j = \begin{cases} (-1)^{d(\beta,j)} & \text{if } \delta \prec j \preceq \beta \\ (-1)^{d(\beta,j)}/2 & \text{if } j \in C_1 \\ (-1)^{d(\gamma,j)} & \text{if } \epsilon \prec j \preceq \gamma \\ (-1)^{d(\gamma,j)}/2 & \text{if } j \in C_2 \end{cases} \tag{11}$$

Example: This shows that $\Lambda = [.5 \quad .5 \quad .5 \quad .5 \quad -1]$ is not 2-MITEable but $\Lambda = [.5 \quad -.5 \quad .5 \quad -.5 \quad 1]$ is.
Since there are only few graphs to be compared with, Λ can be shown to be 2-MITEable or otherwise in this case by arranging them according to the expressions in Eq. (10) and (11).

VII. Conclusion

In this paper, the importance of 2-MITE networks is shown by the fact that the D–stability of these networks is guaranteed if the input-connectivity matrix is nonsingular and has a positive diagonal. A graph-theoretic approach to the problem of synthesis using 2-MITEs is taken. An expression for the powers obtained in a 2-MITE POPL network is arrived at using the theory of Coates graphs and is shown in terms of distances between two vertices in a digraph. This leads to necessary conditions and, for the single-output case, sufficient conditions for a power matrix to be 2-MITEable.

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